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# Some Convergence Theorems for Henstock-Kurzweil Integrable Functions with Values in Riesz Spaces Defined on Euclidean Spaces $\Re^n$

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**Abstract**

Some convergence Theorems for Henstock Integrable functions from Euclidean space $\Re^n$ into Riesz spaces are constructed by using decreasing net of double sequences. We give more general results than those of the convergence Theorems for Henstock Integrable functions with values in sequence spaces.

**Key words:** *double sequence spaces, Henstock integrable functions, Riesz spaces.*

## 1. Introduction

The Henstock-Kurzweil integral for Riesz-space-valued functions defined on bounded subintervals of the real line and with respect to operator-valued measures was investigated in [9] and[10], with respect to $(D)$- convergence (that is a kind of convergence in which the $\varepsilon$-technique is replaced by a technique involving double sequences , see [11] and [12], with respect to the order convergence, see [5] and in [6] with respect to the order convergence but the Henstock-Kurzweil integral for Riesz-space-valued functions was defined on unbounded subintervals of the real line.

The Henstock-Kurzweil integral for real-valued functions defined on Euclidean space $\Re^n$ with respect to volume $\alpha$ was investigated in [15] and [7] and The Henstock-Kurzweil integral for bounded-sequence-space-valued functions defined on Euclidean space $\Re^n$ with respect to volume $\alpha$ was investigated in [1], [2] and [13]. The last work was conducted by constructing The Henstock-Kurzweil integral for Riesz-space-valued functions defined on Euclidean space $\Re^n$ with respect to volume $\alpha$, by using decreasing nets, see [3].

The main goal of this paper is to build the convergence theorems for the Henstock-Kurzweil integral for Riesz-space-valued functions defined on Euclidean space $\Re^n$ with respect to volume $\alpha$, by generalizing the results in the Henstock-Kurzweil integral for bounded-sequence-space-valued functions defined on Euclidean space $\Re^n$ with respect to volume $\alpha$.

## 2. Preliminary

Let $\mathbb{N}$ be the set of all strictly positive integers, $\Re$ the set of the real numbers, $\Re^+$ be the set of all strictly positive real numbers. Moreover, we refer to [16] about the notions of cell, segmentation, partition, $\alpha$-volume, and $\delta$- fine Perron partition.

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*Definition 1* [14].
A Riesz space $L$ is said to be Dedekind complete if every nonempty subset of $L$, bounded from above, has supremum in $L$.

*Definisi 2* [11].
A bonded double sequence $(a_{i,j})_{i,j} \in L$ is called <u>regulator</u> or $(D)$-sequence if, for each $i \in \mathbb{N}, a_{i,j} \downarrow 0$, that is $a_{i,j} \geq a_{i,j+1} \forall j \in \mathbb{N}$ and $\bigwedge_{j\in\mathbb{N}} a_{i,j} = 0$.

*Definition 3* [6].
Given a sequence $(r_n)_n \in L$. Sequence $(r_n)_n$ is said to be $(D)$-convergence to an element $r \in L$ if there exist a regulator $(a_{i,j})_{i,j}$, satisfying the following condition: for every mapping $\rho : \mathbb{N} \to \mathbb{N}$, denoted by $\rho \in \mathbb{N}^{\mathbb{N}}$, there exists an integer $n_0$ such that $|r_n - r| \leq \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$ for all $n \geq n_0$. In this case, the notation is denoted by $(D)\lim_n r_n = r$.

*Definition 4* [6].
A Riesz Space $L$ is said to be weakly $\sigma$-distributive if for every $(D)$- sequence $(a_{i,j})$, then

$$\bigwedge_{\rho\in\mathbb{N}^{\mathbb{N}}} \left( \bigvee_{i=1}^{\infty} a_{i,\rho(i)} \right) = 0 .$$

Throughout the paper, we shall always assume that $L$ is Dedekind complete weakly $\sigma-$distributive Riesz space.

In the principle, this integral is a generalization of Henstock-Kurzweil integral for Riesz-valued functions defined on subintervals of the real line by changing the length of $[a,b] \subset \Re$ with the general volume $\alpha$ of a cell $A \subset \Re^n$, see [15] and [1]. Remember that the volume $\alpha$ on cell $A \subset \Re^n$ is an additive and non negative function from $\Im(A)$ into $\Re$, where $\Im(A)$ is a collection of all subcells in $A$.

Here are some recent results of the Henstock-Kurzweil integral for Riesz-space-valued functions defined on Euclidean space $\Re^n$ with respect to volume $\alpha$.

*Definition 5* [3].
Let $\alpha$ be a volume on $\Re^n$ and $A \subset \Re^n$ be a cell. A function $\overline{f} : \Re^n \to L$ is said to be Henstock-Kurzweil integrable on $A$ with respect to $\alpha$, denoted by $\overline{f} \in HK(A, L, \alpha)$, if there exists an element $\Xi \in L$ and $(D)$-sequence $(a_{i,j})_{i,j} \in L$ such that for every $\rho \in \mathbb{N}^{\mathbb{N}}$ we can find a function $\delta : E \to \Re^+$ such that

$$\left\| P\sum \overline{f}(\overline{x})\alpha(I) - \Xi \right\| = \left\| \sum_{k=1}^{r} \overline{f}(\overline{x}_k)\alpha(I_r) - \Xi \right\| < \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

for every $\delta$-fine Perron partition $P = \{(I, \overline{x})\} = \{(I_1, \overline{x}_1), (I_2, \overline{x}_2), ..., (I_r, \overline{x}_r)\}$ on $A$.

We note that the Henstock-Kurzweil integral with respect to $\alpha$ is well- defined, that is there exists at most one element $\Xi$, satisfying Definition 5 and in this case we have $(HK)\int_A \overline{f} d\alpha = \Xi$. The uniqueness is given in the following theorem.

*Theorem 1* [3].

Let $\alpha$ be a volume on $\Re^n$ and $A \subset \Re^n$ be a cell. If function $\overline{f} \in HK(A, L, \alpha)$, then its $\alpha$-integral is unique.

*Theorem 2* [3].

If $\overline{f}_1, \overline{f}_2 \in HK(A, L, \alpha)$ and $k_1, k_2 \in \Re$, then $k_1\overline{f}_1 + k_2\overline{f}_2 \in HK(A, L, \alpha)$ and $(HK)\int_A \left(k_1\overline{f}_1 + k_2\overline{f}_2\right)d\alpha = k_1(HK)\int_A \overline{f}_1 d\alpha + k_2(HK)\int_A \overline{f}_2 d\alpha$.

*Theorem 3* [3].

If $\overline{f}, \overline{g} \in HK(A, L, \alpha)$ and $\overline{f}(\overline{x}) \le \overline{g}(\overline{x})$ for every $\overline{x} \in A$, then $(HK)\int_A \overline{f} d\alpha \le (HK)\int_A \overline{g} d\alpha$.

*Definition 6* (Elementary Set).

A set $A \subset \Re^n$ which is union of finite cells is called an elementary set.

Every elementary set can be segmented into non-overlapping cells. If $A_1$ and $A_2$ are elementary sets then $A_1 \cup A_2$ and $\overline{A_1 \setminus A_2}$ are also elementary sets. Integration on elementary set can be constructed through the following theorem.

*Teorema 4* [3].

Let $\alpha$ be a volume on $\Re^n$ and $A_1$ and $A_2$ be non-overlapping cells in $\Re^n$ and $A = A_1 \bigcup A_2$. If $\overline{f} \in HK(A_1, L, \alpha)$ and $\overline{f} \in HK(A_2, L, \alpha)$, then $\overline{f} \in HK(A, L, \alpha)$ and

$$(HK)\int_{A = A_1 \bigcup A_2} \overline{f} d\alpha = (HK)\int_{A_1} \overline{f} d\alpha + (HK)\int_{A_2} \overline{f} d\alpha$$

By implementing Theorem 4 and Definition 5 above, we can see immediately that the following holds.

*Corrolary 1* [3].

Given an elementary set $A \subset \Re^n$ and $\alpha$ volume on $A$. A function $\overline{f} : A \to L$ is said to be Henstock-Kurzweil integrable on $A$ with respect to $\alpha$, denoted by $\overline{f} \in HK(A, L, \alpha)$, if $\overline{f} \in HK(A_i, L, \alpha)$ for every i, where $A = \bigcup_{i=1}^{p} A_i$ and $\{A_1, A_2, ..., A_p\}$ is any division on $A$. The Henstock-Kurzweil integral of function $f$ on $A$ is

$$(HK)\int_A \overline{f} d\alpha = \sum_{i=1}^{p} \int_{A_i} \overline{f} d\alpha .$$

*Theorem 5* [3].

A function $\overline{f} : A \to L$ is Henstock-Kurzweil integrable if and only if there exists a $(D)-$sequence $\left(a_{i,j}\right)_{i,j}$ in $L$ such that, for every $\rho \in \mathbb{N}^{\mathbb{N}}$ we can find a function $\delta : A \to \Re^+$ and for every $\delta-$fine Perron partition $P_1 = \{(I, \overline{x})\}$ and $P_2 = \{(I, \overline{x})\}$ on $A$, we have

$$\left| P_1 \sum \overline{f}(\overline{x})\alpha(I) - P_2 \sum \overline{f}(\overline{x})\alpha(I) \right| \le \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

*Theorem 6* [3].

Let $\alpha$ be a volume on a cell $A \subset \Re^n$. If $\bar{f} \in HK(A, L, \alpha)$, then $\bar{f} \in HK(B, L, \alpha)$, for every cell $B \subset A$.

Based on Theorem 6, we define primitif function of Henstock-Kurzweil integrable function $\bar{f}$ on a cell $A \subset \Re^n$ with respect to a volume $\alpha$ as follows.

*Definition 7* [3].

If $\bar{f} \in HK(A, L, \alpha)$ and $\Im(A)$ is a collection of all subcells in $A$, then a function $\bar{F} : \Im(A) \to L$ satisfying

$$\bar{F}(I) = (HK)\int_I \bar{f}\, d\alpha \quad \text{and} \quad \bar{F}(\phi) = \bar{0}$$

for every cell $I \in \Im(A)$ is called $\alpha$ - Primitif of Henstock-Kurzweil integrable function $f$ on $\Im(A)$.

## 3. Main Results and Discussion

Convergence of a sequence of functions is always associated with its limit and function property in the sequence. Let $\{\bar{f}^{(n)}\}$ be a sequence function defined on $I \subset R^n$. Then this sequence is said to be convergent to a function $\bar{f}$ on $I$ if $\lim_{n\to\infty} \bar{f}^{(n)}(\bar{x}) = \bar{f}(\bar{x})$ for every $\bar{x} \in I$. A sequence of functions $\{\bar{f}^{(n)}\}$ is called increasingly monoton or decreasingly monoton if $\bar{f}^{(n)}(\bar{x}) \le \bar{f}^{(n+1)}(\bar{x})$ or $\bar{f}^{(n)}(\bar{x}) \ge \bar{f}^{(n+1)}(\bar{x})$ for every $\bar{x} \in I$, respectively. Let $\bar{f}^{(n)} : I \to L$ for every $n \in N$. Then a sequence function $\{\bar{f}^{(n)}\}$ is convergent to a function $\bar{f}$ at $\bar{x} \in I$ if and only if there exists $(D)$-sequence $(a_{i,j})_{i,j} \in L$ such that for every $\rho \in \square^{\square}$,

$$\left\| \bar{f}^{(n)}(\bar{x}) - \bar{f}(\bar{x}) \right\| < \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

for every $n \ge n_0$. A sequence function $\{\bar{f}^{(n)}\}$ is convergent to a function $\bar{f}$ on $I$ if and only if sequence $\{\bar{f}^{(n)}(\bar{x})\}$ is convergent to a function $\bar{f}(\bar{x})$ for every $\bar{x} \in I$. A sequence function $\{\bar{f}^{(n)}\}$ is uniformly convergent to a function $\bar{f}$ on $I$ if and only if if there exists $(D)$-sequence $(b_{i,j})_{i,j} \in L$ such that for every $\rho \in \square^{\square}$,

$$\left\| \bar{f}^{(n)}(\bar{x}) - \bar{f}(\bar{x}) \right\| < \bigvee_{i=1}^{\infty} b_{i,\rho(i)}$$

for every $n \ge n_0$, $\bar{x} \in I$.

Now, we give some convergence theorems for Henstock Integrable functions with values in $L$.

*Theorem 7.*

(Uniformly Convergence Theorem). Let $I \subset R^n$ be a cell and $\left\{\bar{f}^{(n)}\right\} \subset HK(A,L,\alpha)$ . If sequence of functions $\left\{\bar{f}^{(n)}\right\}$ is convergent uniformly to a function $\bar{f}$ on $I$ , then $\bar{f} \in HK(A,L,\alpha)$ and

$$(H)\int_I \bar{f}\,d\alpha = \lim_{n\to\infty}(H)\int_I \bar{f}^{(n)}d\alpha$$

*Proof:*

A sequence function $\left\{\bar{f}^{(n)}\right\}$ is uniformly convergent to a function $\bar{f}$ on $I$ if and only if if there exists $(D)$-sequence $\left(a_{i,j}\right)_{i,j} \in L$ such that for every $\rho \in \mathbb{N}^{\mathbb{N}}$ ,

$$\left\|\bar{f}^{(n)}(\bar{x}) - \bar{f}(\bar{x})\right\| < \tfrac{1}{4\alpha(I)}\bigvee_{i=1}^{\infty} a_{i,\rho(i)} \qquad (1)$$

for every $n \geq n_0, \bar{x} \in I$ .

A function $f^{(n)} \in HK(A,L,\alpha)$ if there exists $(D)$-sequence $\left(b_{i,j}\right)_{i,j} \in L$ such that for every $\rho \in \mathbb{N}^{\mathbb{N}}$ we can find a function $\delta : E \to \Re^{+}$ such that

$$\left|\sum_{k=1}^{r} f^{(n)}(\bar{x}_k)\alpha(I_r) - \int_I f^{(n)}d\alpha\right| \leq \tfrac{1}{4\alpha(I)}\bigvee_{i=1}^{\infty} b_{i,\rho(i)} \qquad (2)$$

for every $\delta$-fine Perron partition $P = \left\{(I,\bar{x})\right\} = \left\{(I_1,\bar{x}_1),(I_2,\bar{x}_2),...,(I_r,\bar{x}_r)\right\}$ on $A$. And if $P_1 = \{(\bar{x}, I)\}, P_2 = \{(\bar{x}, I)\}$ are $\delta_n$- fine Perron partition on $I$ , then

$$\left\|P_1\sum f^{(n)}(\bar{x})\alpha(I) - P_2\sum f^{(n)}(\bar{x})\alpha(I)\right\| < \tfrac{1}{4}\bigvee_{i=1}^{\infty} c_{i,\rho(i)} \ . \qquad (3)$$

Based on (1) and (3), we have

$$\begin{aligned}\left\|P_1\sum \bar{f}(\bar{x})\alpha(I) - P_2\sum \bar{f}(\bar{x})\alpha(I)\right\| &\leq \left\|P_1\sum \bar{f}(\bar{x})\alpha(I) - P_1\sum \bar{f}^{(m)}(\bar{x})\alpha(I)\right\| + \\ &\quad \left\|P_1\sum \bar{f}^{(m)}(\bar{x})\alpha(I) - P_2\sum \bar{f}^{(m)}(\bar{x})\alpha(I)\right\| + \\ &\quad \left\|P_2\sum \bar{f}(\bar{x})\alpha(I) - P_2\sum \bar{f}^{(m)}(\bar{x})\alpha(I)\right\| \qquad (4) \\ &\leq \frac{\bigvee_{i=1}^{\infty} a_{i,\rho(i)}}{4\alpha(I)}\alpha(I) + \frac{\bigvee_{i=1}^{\infty} c_{i,\rho(i)}}{4} + \frac{\bigvee_{i=1}^{\infty} a_{i,\rho(i)}}{4\alpha(I)}\alpha(I) \\ &< \tfrac{1}{2}\left(\bigvee_{i=1}^{\infty} d_{i,\rho(i)}\right)\end{aligned}$$

where $d_{i,\rho(i)} = a_{i,\rho(i)} + c_{i,\rho(i)}$. This shows that $\bar{f} \in HK(A,L,\alpha)$. Furthermore, since $\bar{f} \in HK(A,L,\alpha)$, then there exists $(D)$-sequence $\left(b_{i,j}\right)_{i,j} \in L$ such that for every $\rho \in \mathbb{N}^{\mathbb{N}}$ we can find a function $\delta' : E \to \Re^{+}$ such that

$$\left|\sum_{k=1}^{r} f^{(n)}(\bar{x}_k)\alpha(I_r) - \int_I f^{(n)}d\alpha\right| \leq \tfrac{1}{4\alpha(I)}\bigvee_{i=1}^{\infty} b_{i,\rho(i)} \qquad (5)$$

for every $\delta$ -fine Perron partition $P'=\{(I,\bar{x})\}$ on $A$. By taking $\delta(\bar{x})=\min\{\delta'(\bar{x}),\delta_n(\bar{x})\}$ for every $\bar{x}\in I$ and if $P=((\bar{x},I))$ is arbitrary $\delta$ - fine Perron partition on $I$ , then we have

$$\begin{aligned}\left\|(H)\int_I \bar{f}\,d\alpha-(H)\int_I \bar{f}^{(n)}d\alpha\right\| &\le \left\|(H)\int_I \bar{f}\,d\alpha-P\sum \bar{f}\,\alpha(I)\right\|+\\ &\quad \left\|P\sum \bar{f}\,\alpha(I)-P\sum \bar{f}^{(n)}\alpha(I)\right\|+ \qquad (6)\\ &\quad \left\|P\sum \bar{f}^{(n)}\alpha(I)-(H)\int_I \bar{f}^{(n)}d\alpha\right\|\\ &\le \frac{\bigvee_{i=1}^{\infty} b_{i,\rho(i)}}{4\alpha(I)}\alpha(I)+\frac{\bigvee_{i=1}^{\infty} a_{i,\rho(i)}}{4\alpha(I)}\alpha(I)+\frac{\bigvee_{i=1}^{\infty} c_{i,\rho(i)}}{4}\\ &< \frac{1}{4}\bigvee_{i=1}^{\infty} e_{i,\rho(i)}\end{aligned}$$

where $e_{i,\rho(i)}=a_{i,\rho(i)}+b_{i,\rho(i)}+c_{i,\rho(i)}$. Thus, we have proved that

$$(H)\int_I \bar{f}\,d\alpha=\lim_{n\to\infty}(H)\int_I \bar{f}^{(n)}d\alpha\,.$$

*Theorem 8* (Monoton Convergence Theorem).

Let $I\subset R^n$ be a cell and $\{\bar{f}^{(n)}\}\subset HK(A,L,\alpha)$. If sequence of monoton functions $\{\bar{f}^{(n)}\}$ is convergent to a function $\bar{f}$ on $I$ and $\lim_{n\to\infty}(H)\int_I \bar{f}^{(n)}d\alpha$ exists, then $\bar{f}\in HK(A,L,\alpha)$ and

$$(H)\int_I \bar{f}\,d\alpha=\lim_{n\to\infty}(H)\int_I \bar{f}^{(n)}d\alpha$$

*Proof*:

We just need to prove for sequence $\{\bar{f}^{(n)}\}$ of increasingly monoton on $I$. Based on assumption , we can find $\Xi\in L$ so that $\lim_{n\to\infty}(H)\int_I \bar{f}^{(n)}d\alpha=\Xi$. Since $\{\bar{f}^{(n)}\}$ of increasingly monoton on $I$ , it follows that this sequence, $\left\{(H)\int_I \bar{f}^{(n)}d\alpha\right\}$ is increasingly monoton and $\bar{a}$ as its least upper bound. Hence, there exists $(D)$-sequence $(a_{i,j})_{i,j}\in L$ such that for every $\rho\in\mathbb{N}^{\mathbb{N}}$ ,there exists $n_0\in N$ , such that if $n\ge n_0$, we obtain

$$\left\|\Xi-(H)\int_I \bar{f}^{(n)}d\alpha\right\|<\frac{\bigvee_{i=1}^{\infty} a_{i,\rho(i)}}{4} \qquad (7)$$

Since $\{\bar{f}^{(n)}\}$ is convergent to a function $\bar{f}$ on $I$ , then there exists $(D)$-sequence $(b_{i,j})_{i,j}\in L$ such that for every $\rho\in\mathbb{N}^{\mathbb{N}}$ and $\bar{x}\in I$ there exists $m_0=m_0(\varepsilon,\bar{x})$ such that if $n\ge m_0$, we have

$$\left\| \overline{f}^{(n)}(\overline{x}) - \overline{f}(\overline{x}) \right\| < \frac{\bigvee_{i=1}^{\infty} b_{i,\rho(i)}}{4\alpha(I)+1} \tag{8}$$

Since $\overline{f}^{(n)} \in HK(A,L,\alpha)$ for every $n$, it follows that then there exists $(D)$-sequence $(c_{i,j})_{i,j} \in L$ such that for every $\rho \in \mathbb{N}^{\mathbb{N}}$, there exists a positive function $\delta_n : I \to (0,\infty)$ such that for every $\delta_n$-fine Perron partition $P_n = ((\overline{x}, I))$ on $I$, we have

$$\left\| P_n \sum \overline{f}(\overline{x})\alpha(I) - \overline{F}^{(n)}(I) \right\| < \frac{\bigvee_{i=1}^{\infty} c_{i,\rho(i)}}{2^{n+2}} \tag{9}$$

Taking positive function $\delta : I \to (0,\infty)$ where $\delta(\overline{x}) = \delta_{m(\varepsilon,\overline{x})}(\overline{x})$ for every $\overline{x} \in I$ and $m(\varepsilon,\overline{x}) = \max\{n_0, m_0(\varepsilon,\overline{x})\}$. Hence, if $P = ((\overline{x}, I))$ is $\delta$-fine Perron partition on $I$, then

$$\begin{aligned}
\left\| P\sum \overline{f}(\overline{x})\alpha(I) - \Xi \right\| &= \left\| \sum_{i=1}^{k} \overline{f}(\overline{x}_i)\alpha(I_i) - \Xi \right\| \\
&\le \left\| \sum_{i=1}^{k} \left\{ \overline{f}(\overline{x}_i)\alpha(I_i) - \overline{f}^{(m(\varepsilon,\overline{x}_i))}(\overline{x}_i)\alpha(I_i) \right\} \right\| + \\
&\quad \left\| \sum_{i=1}^{k} \left\{ \overline{f}^{(m(\varepsilon,\overline{x}_i))}(\overline{x}_i)\alpha(I_i) - (H)\int_{I_i} \overline{f}^{(m(\varepsilon,\overline{x}_i))} d\alpha \right\} \right\| \\
&\quad + \left\| \sum_{i=1}^{k} (H)\int_{I_i} \overline{f}^{(m(\varepsilon,\overline{x}_i))} d\alpha - \Xi \right\| \qquad (10) \\
&< \frac{\bigvee_{i=1}^{\infty} b_{i,\rho(i)}}{4\alpha(I)} \sum_{i=1}^{k} \alpha(I_i) + 2\sum_{n=1}^{\infty} \frac{\bigvee_{i=1}^{\infty} c_{i,\rho(i)}}{2^{n+2}} + \frac{\bigvee_{i=1}^{\infty} a_{i,\rho(i)}}{4} \\
&< \bigvee_{i=1}^{\infty} f_{i,\rho(i)}
\end{aligned}$$

where, $f_{i,\rho(i)} = \bigvee_{i=1}^{\infty} a_{i,\rho(i)} + \bigvee_{i=1}^{\infty} b_{i,\rho(i)} + \bigvee_{i=1}^{\infty} c_{i,\rho(i)}$.

This shows that $\overline{f} \in HK(A,L,\alpha)$ and $\lim_{n\to\infty} (H)\int_I \overline{f}^{(n)} d\alpha = (H)\int_I \overline{f} d\alpha = \overline{a}$.

Furthermore, Let $\inf\{\overline{f}^{(n)}\} = \{\inf(f_1^{(n)}), \inf(f_2^{(n)}), \ldots\}$ be infimum of sequence $\{\overline{f}^{(n)}\}$.

*Theorem 9.*

Let $I \subset R^n$ be a cell. If $\overline{f}^{(n)}, \overline{g}_n \in HK(A,L,\alpha)$ and $\overline{f}^{(n)} \ge \overline{g}$ for every $n$, then $\inf\{\overline{f}^{(n)}\} \in HK(A,L,\alpha)$

*Proof*:

Let $\overline{h}^{(n)}(\overline{x}) = \min\{\overline{f}^{(1)}(\overline{x}), \overline{f}^{(2)}(\overline{x}), \ldots, \overline{f}^{(n)}(\overline{x})\}$ be defined for every $\overline{x} \in I$. It follows that $\overline{h}^{(n)}(\overline{x}) \ge \overline{g}(\overline{x})$ for every $\overline{x} \in I$. Thus, we have a decreasingly bounded sequence $\{\overline{h}^{(n)}\}$

where $\inf\left\{\bar{f}^{(n)}\right\} = \bar{h}(\bar{x}) \geq \bar{g}(\bar{x})$ be its bound for every $\bar{x} \in I$. Since $\bar{f}^{(n)} \in HK(A,L,\alpha)$ for every $n$, then $h^{(n)} \in HK(A,L,\alpha)$ and since $\bar{g} \leq \bar{h}^{(n)} \leq \bar{f}^{(k)}$ for every $n \geq k$, it follows that

$$(H)\int_I \bar{g}d\alpha \leq (H)\int_I \bar{h}^{(n)}d\alpha \leq (H)\int_I \bar{f}^{(k)}d\alpha \qquad (11)$$

and $\lim_{n\to\infty}(H)\int \bar{h}^{(n)}d\alpha$ exists, say $\bar{a} = \lim_{n\to\infty}(H)\int \bar{h}^{(n)}d\alpha$. Thus,

$$(H)\int_I \bar{g}d\alpha \leq \bar{a} \leq (H)\int_I \bar{f}^{(k)}d\alpha \text{ for every } k.$$

And, from Theorem 2, this shows that $\inf\left\{\bar{f}^{(n)}\right\} \in HK(A,L,\alpha)$.

*Theorem 10.* (Fatou Lemma).

Let $I \subset R^n$ be a cell and $\bar{f}^{(n)} \in HK(A,L,\alpha)$ for every $n$.

If $\bar{f} = \lim_{n\to\infty} \inf_{n\geq k}\left\{\bar{f}^{(k)}\right\}$ and $\bar{f} = \lim_{n\to\infty} \inf_{n\geq k}\left\{(H)\int \bar{f}^{(n)}d\alpha\right\} < +\infty$, then $f \in HK(A,L,\alpha)$ and

$$(H)\int_I \bar{f}d\alpha \leq \lim_{n\to\infty}(H)\int_I \bar{f}^{(n)}d\alpha$$

*Proof*:

Let $\bar{h}^{(n)}(\bar{x}) = \inf_{k\geq n}\left\{\bar{f}^{(k)}\right\}$ be defined for every $\bar{x} \in I$. It follows that $\left\{\bar{h}^{(n)}\right\}$ is increasingly sequence on $I$ and $\bar{h}^{(n)}(\bar{x}) \leq \bar{f}^{(n)}(\bar{x})$ for every $\bar{x} \in I$. Thus,

$$\lim_{n\to\infty}(H)\int_I \bar{h}^{(n)}d\alpha \leq \lim_{n\to\infty}(H)\int_I \bar{f}^{(n)}d\alpha \qquad (12)$$

and $\left\{\bar{h}^{(n)}\right\}$is convergent to $\bar{f}$. Since $\lim_{n\to\infty}(H)\int \bar{h}^{(n)}d\alpha$ exists, then under Monoton convergence Theorem, we have $\bar{f} \in HK(A,L,\alpha)$ and

$$(H)\int_I \bar{f}d\alpha = \lim_{n\to\infty}(H)\int \bar{h}^{(n)}d\alpha \leq \lim_{n\to\infty} \inf_{n\geq k}\left\{(H)\int \bar{f}^{(n)}d\alpha\right\} \qquad (13)$$

*Corrolary 3.5.*

Let $I \subset R^n$ be a cell, $\bar{f}^{(n)}, g_n \in HK(A,L,\alpha)$ and $-\bar{g} \leq \bar{f}^{(n)} \leq \bar{g}$ for every $n$. If $\lim_{n\to\infty}\bar{f}^{(n)} = \bar{f}$ then $f \in HK(A,L,\alpha)$ and $(H)\int_I \bar{f}d\alpha = \lim_{n\to\infty}(H)\int_I \bar{f}^{(n)}d\alpha$.

## 4. Concluding Remarks

The convergence theorems for the Henstock-Kurzweil integral for Riesz-space-valued functions defined on Euclidean space $\Re^n$ with respect to volume $\alpha$, have been built by generalizing the results in the Henstock-Kurzweil integral for bounded-sequence-space-valued functions defined on Euclidean space $\Re^n$ with respect to volume $\alpha$. Further works are to compare the results with McShane integrable functions especially with srong McShane integrable functions.

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